Introduction

This is the first of a a list of "preamble" assignments, to help you learn, review, or solidify necessary background knowledge for Probability this Fall.

If you don't have time this summer, you could wait to learn this material until we need it during the term. However, if you have the time, I'd recommend working through it before term starts. This will make it easier to keep up during the term, especially since we are on a condensed 7-week schedule.

I've included brief answers or hints at the end, but not full solutions. If you have any questions, feel free to get in touch. I'm available all summer.

Sets and set operations

The language of sets and set operations is particularly important in probability. I will start by giving some basic examples of how these are used in probability theory, without explaining everything. Look over the examples for now, without worrying if there are unfamiliar terms, just to get a sense of what we need sets and set operations for. Then do the reading, which will explain all the unfamiliar terms, and do the exercises, which will help show how they are used.

NOTE: Set theory has gone in and out of fashion in education. So depending on your school system, you may have seen all of this material in an early grade, or you may never have seen it before. This isn't a reflection of the quality of teaching; it's only a choice of educational philosophy. If this is all familiar, feel free to skip through and just check if anything looks new. (Also, if you've taken Logic and Proofs here at Bennington, then this should all be familiar!) If this is all unfamiliar, please spend some time with it, and feel free to ask me questions.

Examples

Example 1: One die. Suppose we roll one 6-sided die. Then the set of possible outcomes (called the *sample space*) is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

The probability is a function from S to the non-negative real numbers $\mathbb{R}_{\geq 0}$, written

$$P\colon S \to \mathbb{R}_{\geq 0}.$$

The function assigns every point in the sample space the same probability of $1/6$:

$$P(x) = \frac{1}{6} \qquad \text{for every } x \in S.$$

The result that "the number thrown is odd" is called an *event*, and it can be described as a *subset* of the sample space:

$$O = \{1, 3, 5\}$$

The probability of O occuring is $1/2$. We could define other events; for example, the subset

$$L = \{4, 5, 6\}$$

describes the event that "the number rolled is 4 or larger". The subset

$$T = \{2\}$$

describes the event that "the number rolled is a 2".

We can make new events from given ones; for example, the event that "the number rolled is odd *and* is four or larger" is the event "O AND L", which can be described by the set *intersection*

$$O \cap L = \{5\}.$$

Similarly, the event that "the number rolled is odd *or* is four or larger" is the event "O OR L", which can be described by the set *union*

$$O \cup L = \{1, 3, 4, 5, 6\}.$$

(Note that "OR" in math always includes the possibility of both; "O OR L" means O happens, or L happens, or both O and L happen.) That last event could also be described as "the number rolled is *not* 2". This is the same as saying event T does NOT happen, which can be described by the set *complement*

$$\overline{T} = \{1, 3, 4, 5, 6\}.$$

Example 2: Two dice. If we roll two dice, there are 36 possible outcomes, which we can list as ordered pairs. The sample space is the set U,

$$U = \{(1, 1), (1, 2), (1, 3), \ldots, (2, 1), (2, 2), \ldots (6, 6)\}.$$

Note that, for example, $(2, 3)$ and $(3, 2)$ are different outcomes, because we can assume that the dice are distinguishable (it doesn't change anything if we paint one die red and the other one green). This set is called the *Cartesian product* of the set S from the previous example with itself:

$$U = S \times S = S^2.$$

The probability of each outcome is $1/36$:

$$P\colon U \to \mathbb{R}_{\geq 0}, \qquad P(x) = \frac{1}{36} \qquad \text{for all } x \in U.$$

We can make events as before. For example, the event that "the sum of the dice rolled is 7" would be identified with the subset Q of the sample space, where

$$Q = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}.$$

The probability of Q is $6/36$, that is, $1/6$. As before, we could make more events, and combine events in the same way.

Suppose we are interested in the *sum* of the two dice. This could vary from 2 up to 12. The sum of the two dice is a function: we input an element of the sample space U (a possible roll of the two dice), and we output the sum of the two dice. Let's call this function X. Then

$$X\colon U \to \{2, 3, \ldots, 12\}.$$

The function X is called a *random variable.* We could study the sum of the two dice by studying the function X. Alternately, if all we cared about was the sum of the two dice, we could have started out by saying that the set of possible outcomes, our sample space, is the set

$$\{2, 3, \ldots, 12\}.$$

If we did that, the probability of each outcome would no longer be the same; the probability of a sum of 2 is $1/36$, the probability of a sum of 3 is $2/36$ (since it could happen in two ways), and so on.

Example 3: Tossing coins. If we toss one coin, we would have the set of possible outcomes

$$C = \{H, T\},$$

for Heads and Tails, with probability 1/2 for each. If we tossed a coin twice, we would have the sample space

$$C \times C = C^2 = \{(H, H), (H, T), (T, H), (T, T)\},$$

giving four possible outcomes, with probability 1/4 for each. If we tossed the coin ten times, we would have the sample space

$$C^{10} = \{(H, H, H, H, H, H, H, H, H, H), (H, H, H, H, H, H, H, H, H, T), \dots\},$$

with a total of $2^{10} = 1024$ possible outcomes, and a probability of 1/1024 for each one.

As before, we could define various events, and combine events in various ways, depending on what we are interested in. We could also define random variables: for example, when flipping a coin 10 times, we could ask, how many heads are there total?

Example 4: Lengths of bolts. Suppose that we have a machine that is making bolts which are supposed to be exactly one inch long. The machine cannot be perfect, so the bolts cannot be exactly two inches long.[1] Let's say that, in the absolute worst case, the length will never be less than 1.5 inches or greater than 2.5 inches. The length is not restricted to be in a finite set of possibilities, like in the previous examples; it could be anything in this range. Then the set of all possible outcomes for the lengths is the set of all real numbers between 1.5 and 2.5:

$$L = \{x \big| 1.5 \leq x \leq 2.5\}.$$

An alternate symbol for this is the *closed interval*

$$L = [1.5, 2.5].$$

We have to treat probabilities in this case differently; the probability of having a bolt of ANY exact length is 0, because we can never measure anything exactly. Instead, we have to ask, what is the probability of a certain range? For example, to be acceptable, we might need the bolt to be in the range $[1.95, 2.05]$; we could ask what the probability is that it falls in that range. This requires some trickier reasoning (in particular some calculus!), so we'll get back to it later.

READING

We can find everything we need about sets, set operations, and abstract functions, in the book *Book of Proof* by Richard Hammack. I've provided a pdf; you can also find a free pdf by looking up Richard Hammack's site for the book. Not everything he talks about will be necessary for our class; I'll try to indicate this as I go.

For each of the readings, try to go slowly. Write out examples as you read. Make a list of questions, and a running summary. Ask me questions if you are stuck.

[1]What would that even mean? How would we measure if the bolt was *exactly* two inches long?

1. CONCEPT OF SETS; SOME STANDARD SETS; SET-BUILDER NOTATION Hammack 1.1. Read everything except Example 1.2. In particular, work through Example 1.1 (treat it like solved exercises), and check that your answers agree with his.
2. CARTESIAN PRODUCTS Hammack 1.2. This material is important, and can be tricky; read through everything carefully.
3. SUBSETS Hammack 1.3. This is important, but in this class we only need the simplest examples of this concept. Hammack is trying to prepare you for trickier cases of this concept you might see elsewhere in math; so you can skip over his Example 1.6.
4. SET OPERATIONS Hammack 1.5 and 1.6. We mostly only need the case of finite sets; so for Section 1.5, you can read up through Example 1.8, part 12, and stop there. For section 1.6, you can skip Example 1.11.
5. VENN DIAGRAMS Hammack 1.7. This isn't strictly necessary, but you'll find that it helps a lot in reasoning through some questions. I would suggest redrawing each diagram yourself so you understand how they're made.
6. INDEXED SETS Hammack 1.8. This isn't super important for us, but it will come up occasionally. It might be worth reading the first part briefly through, up to the end of Example 1.13, but don't worry too much if it isn't totally clear on a first reading.
7. LOGIC Hammack 2.2 and 2.3. If you aren't so familiar with logical operations, it might be worth taking a read through Hammack 2.2, where he explains AND, OR, and NOT. You might want to glance back to Section 2.1 if his use of the word "statement" is unclear. Also take a read through 2.3, where he explains "if-then" statements; you don't need to follow all this closely, but it will be helpful to get the idea.

EXERCISES

Try the following exercises. I tried to just pick a few, with which you can check your understanding. If you can do those without a problem, you're fine. If you find you need more practice, you can do some of the exercises I haven't assigned. Note that Hammack provides answers to most of the odd-numbered problems at the end of the book.

- Exercises for Section 1.1: # 3, 11, 17, 19
- Exercises for Section 1.2: # 1 (all parts), 9
- Exercises for Section 1.3: # 1, 9
- Exercises for Section 1.5: # 1 (all parts), 3 ((a)–(d) only)
- Exercises for Section 1.6: # 1 (all parts)
- Exercises for Section 1.7: # 1, 3, 5, 7, 11, 13
- Exercises for Section 1.8: # 1, 3
- Exercises for Section 2.2: # 1, 3, 5, 7, 9