Introduction

In this second "preamble" assignment, I introduce (or review) the binomial theorem. On the face of it, this is a fact about algebra, but it's actually deeper than that, and also pops up in many other places.

I won't try to explain now how the binomial theorem will be important in probability—I'll leave that until we get to it in the class. For the time being, you can just treat it this assignment as straight algebra problems.

The binomial theorem will also be important in the next "preamble" assignment, on the natural exponential function.

If you are familiar with the binomial theorem already, you might still want to complete the exercises—the results of the exercises will be used later.

Binomial Theorem: The Problem

In this first section, I want to explain the basic problem that the binomial theorem is trying to solve. In the next section, I'll state the theorem, which is the solution to the problem.

Here's one version of the problem: let's suppose that we want to expand out

$$(a+b)^{100}.$$

How could we do it? (I'll explain reasons why we might *want* to do such a crazy thing later.)

Let me remind you what I mean by "expand". Certainly you've done this before with $(a+b)^2$:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a(a+b)+b(a+b) \\ &= a\cdot a + a\cdot b + b\cdot a + b\cdot b \\ &= a^2+ab+ba+b^2 \\ &= a^2+2ab+b^2.\end{aligned}$$

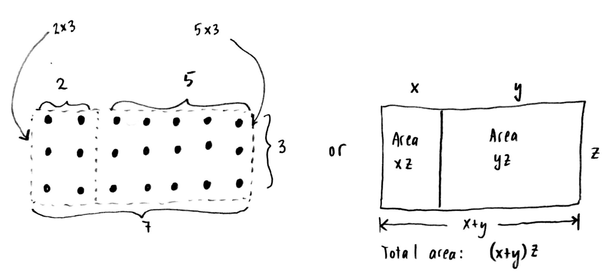
(Above, I've made repeated use of the "distributive property",

$$(x+y)z = xz + yz$$

for any x, y, and z. For example,

$$(2+5)\cdot 3 = 2\cdot 3 + 5\cdot 3:$$

"two threes plus five threes equals seven threes". Or as a diagram:



Just in case that was something not totally clear in the past!)

To expand $(a+b)^3$, you can proceed two ways: one way is to use the answer for $(a+b)^2$, and the distributive property:

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2\\ &= (a+b)\left(a^2+2ab+b^2\right)\\ &= \left(a\cdot a^2 + a\cdot 2ab + a\cdot b^2\right)\\ &\qquad + \left(b\cdot a^2 + b\cdot 2ab + b\cdot b^2\right)\\ &= a^3+2a^2b+ab^2\\ &\qquad + a^2b+2ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

(Organizational trick: note how I've lined up the terms, when I split the lines in the third and fourth steps! This will be helpful when it gets more complicated.)

The other way you can expand $(a+b)^3$ is to note that each term of $(a+b)(a+b)(a+b)$ consists of one choice, of either a or b, for each of the three brackets:

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b)\\ &= a\cdot a\cdot a + a\cdot a\cdot b + a\cdot b\cdot a + a\cdot b\cdot b\\ &\quad + b\cdot a\cdot a + b\cdot a\cdot b + b\cdot b\cdot a + b\cdot b\cdot b\\ &= a^3+a^2b+a^2b+ab^2+a^2b+ab^2+ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3.\end{aligned}$$

(Both methods will be important later, so make sure you understand both ways.)

Now, back to my original problem: how would you expand

$$(a+b)^{100} \quad ?$$

One strategy could be to start expanding smaller powers by hand, and see if there is a pattern we can follow. If you continue the procedure above, you should find that

$$\begin{aligned}(a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\ (a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\\ (a+b)^6 &= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6\\ (a+b)^7 &= a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7\end{aligned}$$

and so on.

EXERCISE 1: Verify my answer for $(a+b)^4$, using both of the methods that I suggested above.

EXERCISE 2: Assuming my answer for $(a+b)^7$ is correct, use it to find the expansion of $(a+b)^8$, using the first method. (Remember the organizational trick to save some work.)

Obviously it will be very tedious to do this up to the 100th power (unless we somehow program a computer to do the work for us). But we can see some patterns in the above. Assuming that the apparent patterns are valid, we could make the following guesses about the expansion of $(a+b)^{100}$:

- The terms should go a^{100}, then $a^{99}b$, then $a^{98}b^2$, then $a^{97}b^3$, with the power of a going down one each time, and the power of b going up one, until we get to b^{100}; each of those terms will have some positive whole number in front (called a "coefficient")
- The powers of a and b should always add to 100 for each term
- The coefficient of a^{100} should be 1, (that is, the term is just a^{100}), and the coefficient of $a^{99}b$ should be 100
- There should be 101 terms in total
- The coefficients should increase up to the middle term (or pair of middle terms), and then decrease
- The coefficients should be symmetric: if we start at the last term and read the coefficients right to left, we get the same sequence of numbers as if we read them from the beginning left to right

Based on the above patterns, the expansion of $(a+b)^{100}$ ought to look something like:

$$\begin{aligned}(a+b)^{100} = a^{100} + 100a^{99}b + (??)a^{98}b^2 + (??)a^{97}b^3 + \\ \cdots + (??)a^2b^{98} + 100ab^{99} + b^{100},\end{aligned}$$

where the (??) represent numbers we still need to figure out.

Assuming all these guesses turn out to be valid (they do), the trickiest part is going to be working out those unknown coefficients. We could do this if we could see the pattern in the coefficients.

It will be convenient to number the coefficients starting at zero. The zeroth coefficient is always 1 (for example, $(a+b)^3 = 1 \cdot a^3 + \cdots$); the first coefficient is just equal to the power of $(a+b)$ that you are expanding (for example, $(a+b)^3 = 1 \cdot a^3 + 3a^2b + \cdots$). But what's the pattern in the following coefficient? Or the one after that?

If you haven't seen this before, I'd recommend trying to find patterns yourself before proceeding:

EXERCISE 3: The second coefficients form the sequence $1, 3, 6, 10, 15, 21, \ldots$ (look at the computations I wrote out above). See if you can find a pattern in the second coefficient of each expansion, as the power increases.[1]

EXERCISE 4: If you got an answer to the previous exercise, see if you can find a pattern in the third coefficient of each expansion, as the power increases: $1, 4, 10, 20, 35, \ldots$

[1] If you have seen this before, you are probably yelling "Pascal's Triangle!!". I'll talk about Pascal's triangle at the end of this assignment. If you know about Pascal's Triangle, using it would save a lot of work. However, note that it would still be quite tedious to compute all the first hundred rows of the triangle to compute $(a+b)^{100}$. I'd like to be able to expand $(a+b)^{100}$ directly.

BINOMIAL THEOREM: THE STATEMENT

The "binomial theorem" is the general solution to the problem I discussed above. That is, it tells you exactly how to expand $(a+b)^n$ for any positive whole number n, without going through the whole process of multiplying out the brackets. In particular, it tells you what the exact pattern is in the coefficients of the expansion.

To build up to the theorem, I should tell you some of the patterns. One way of describing the pattern in the second coefficient $(1, 3, 6, 10, 15, 21, \dots)$ is as follows:

$$\begin{aligned} 1 &= \frac{2 \cdot 1}{2} \\ 3 &= \frac{3 \cdot 2}{2} \\ 6 &= \frac{4 \cdot 3}{2} \\ 10 &= \frac{5 \cdot 4}{2} \\ 15 &= \frac{6 \cdot 5}{2} \\ 21 &= \frac{7 \cdot 6}{2} \\ &\dots \end{aligned}$$

(Check my answers!) That is, the second coefficient when the power is n is equal to

$$\frac{n(n-1)}{2}.$$

With this answer, you might want to go back and try again to get the pattern in the third coefficient...

Did you get it? It's tricky. The third coefficients are $1, 4, 10, 20, 35, \dots$. Here's one way to get them, following a similar pattern:

$$\begin{aligned} 1 &= \frac{3 \cdot 2 \cdot 1}{6} \\ 4 &= \frac{4 \cdot 3 \cdot 2}{6} \\ 10 &= \frac{5 \cdot 4 \cdot 3}{6} \\ 20 &= \frac{6 \cdot 5 \cdot 4}{6} \\ 35 &= \frac{7 \cdot 6 \cdot 5}{6} \\ &\dots \end{aligned}$$

Why 6 in the bottom? That seems weird. One way to make it seem more natural is to write 6 as $3 \cdot 2 \cdot 1$, for example:

$$10 = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1}$$

EXERCISE 5: Write the formula for the third coefficient, when the power is n.

EXERCISE 6: Try to guess the pattern for the fourth coefficient. Check with the computations from before to make sure your guess gives you the right answers. Also, write the formula for the fourth coefficient, when the power is n.

Here is the theorem:

BINOMIAL THEOREM: Suppose that a and b are any real numbers, and that n is a positive whole number. Then

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2\cdot 1}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}a^{n-2}b^2 \\ &+ \frac{n(n-1)(n-2)(n-3)}{4\cdot 3\cdot 2\cdot 1}a^{n-3}b^3 + \cdots \\ &+ \frac{n(n-1)(n-2)(n-3)\cdots 3\cdot 2}{(n-1)\cdots 3\cdot 2\cdot 1}ab^{n-1} \\ &+ \frac{n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1}{n(n-1)\cdots 3\cdot 2\cdot 1}b^n.\end{aligned}$$

□

Note that the terms after the "$\cdots$" tell you when to stop writing terms (you stop when you get to b^n).

I'll come back to explaining *why* the binomial theorem works later.

Here are some exercises to help you understand the statement of the binomial theorem, and to help you apply it.

EXERCISE 7: Use the binomial theorem to write out the expansion of $(a+b)^4$. Check that you get the same answer as before.

EXERCISE 8: Use the binomial theorem to write out the expansion of $(a+b)^5$, $(a+b)^6$, $(a+b)^7$, and $(a+b)^8$. Check that you get the same answers as before. (NOTE: When you do the computations, I strongly recommend that you do them by hand. When you simplify a fraction, do not multiplying out the entire top, and multiply out the entire bottom, and then divide; rather, make as many cancellations as you can first to simplify the fraction.)

EXERCISE 9: We have noted that the pattern of coefficients is symmetric: for a given $(a+b)^n$, if we start from the last term and read the coefficients right to left, we get the same sequence as if we start at the first term and go right to left. For example, for $(a+b)^7$, the sequence is $1, 7, 21, 35, 35, 21, 7, 1$. Explain why! (There are at least two answers: one has to do with cancellation in the fractions. The other goes back to what $(a+b)^n$ means. Find both explanations if you can!)

EXERCISE 10: Write out the first five terms of the expansion of $(a+b)^{100}$. Simplify the coefficients as much as you can. (It's OK to use a calculator if you like.)

EXERCISE 11: Suppose x is any number, and n is any positive whole number. Expand

$$(1+x)^n.$$

(Your answer will be a general formula, similar to the statement of the binomial theorem itself.)

Here are some exercises which give results that we will be using later. (They will be used both in the next "preamble" assignment, and also later in the class.)

EXERCISE 12: Expand each of the following:
(a) $\left(1+\frac{1}{2}\right)^2$
(b) $\left(1+\frac{1}{3}\right)^3$
(c) $\left(1+\frac{1}{4}\right)^4$

EXERCISE 13: Suppose n is any positive whole number. Expand the following:

$$\left(1+\frac{1}{n}\right)^n.$$

(Your answer will be a general formula, similar to the statement of the binomial theorem itself.)

EXERCISE 14: Suppose that x is any number, and n is any positive whole number. Expand the following:

$$\left(1+\frac{x}{n}\right)^n.$$

EXERCISE 15: Suppose that n is any positive whole number. Expand the following:

$$\left(1-\frac{1}{n}\right)^n.$$

EXERCISE 16: Algebraically simplify your answer to Exercise 13. Do it in such a way that n only appears in the denominators of any fractions: eliminate all n from the numerators. (I will give the answer to this problem in the next assignment.)

EXERCISE 17: Same as Exercise 16, except simplify the answers to Exercises 14 and 15.

BINOMIAL THEOREM: THE PROOF

Why is the binomial theorem true? There are many ways to explain it. The explanation I give here will also be important for probability problems we will study later.

I will first give the explanation for a specific n (let's say $n = 4$). The problem is to expand

$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b).$$

Each term in the answer will correspond to picking either a or b in each bracket. For example, if I pick b in the first two brackets, and a in the following two brackets, I have the term

$$b \cdot b \cdot a \cdot a.$$

Now, there is only one term that produces a^4, the one which picks a for all four brackets: $a^4 = a \cdot a \cdot a \cdot a$. So the expansion starts with a^4.

But there are *four* terms which produce a^3b. They are

$$a \cdot a \cdot a \cdot b, \quad a \cdot a \cdot b \cdot a, \quad a \cdot b \cdot a \cdot a, \quad \text{and} \quad b \cdot a \cdot a \cdot a.$$

I am picking a in three of the brackets, and b in the fourth. There are 4 choices for which bracket I choose to be a b. Hence, the next term in the expansion is $4a^3b$.

There are *six* terms which produce a^2b^2. They are

$$a \cdot a \cdot b \cdot b, \quad a \cdot b \cdot a \cdot b, \quad a \cdot b \cdot b \cdot a, \quad b \cdot a \cdot a \cdot b, \quad b \cdot a \cdot b \cdot a, \quad \text{and} \quad b \cdot b \cdot a \cdot a.$$

Hence, the next term in the expansion is $6a^2b^2$.

But, how can I find the number 6 more systematically (rather than just listing all possibilities)?

I must choose 2 of the brackets to be b (and the others will be a). There are 4 choices for the first bracket to choose to be a b; then there are 3 choices remaining for the second bracket to be a b. So it would seem we would have

$$4 \times 3 = 12$$

choices. But this is incorrect, because we have over-counted. For example, I thought of choosing the first bracket, then the second bracket, as one choice; but I thought of choosing the second bracket, then the first bracket, as another choice. However, those both give

$$b \cdot b \cdot a \cdot a.$$

It doesn't matter what order I pick the two brackets to be b. So I have over-counted by a factor of 2. The correct number of ways to pick 2 brackets out of 4 to be b is

$$\frac{4 \times 3}{2}.$$

Hence the next term in the expansion is $6a^2b^2$.

The terms which give ab^3 correspond to choosing 3 of the brackets to be b. If I choose them in order, there are

$$4 \times 3 \times 2$$

ways to do it. However, it doesn't matter the order; so I have over-counted by the number of ways I could put three things in order. That over-counting factor is

$$3 \times 2 \times 1,$$

so the correct number of ways to choose 3 brackets for b is

$$\frac{4 \times 3 \times 2}{3 \times 2 \times 1} = 4.$$

Hence the next term is $4ab^3$.

(Note that I could have done this more easily by just picking one bracket to be an a. But I wanted to illustrate the general procedure.)

Now I will repeat this reasoning for any positive whole number exponent n.

PROOF OF THE BINOMIAL THEOREM: Suppose a and b are any numbers, and n is any positive whole number. The expansion of

$$(a+b)^n = \underbrace{(a+b)(a+b)\cdots(a+b)}_{n \text{ factors}}$$

consists of one term for every possible choice of either an a or a b from each bracket.

Let k be any non-negative whole number less than or equal to n. Then there will be some terms which simplify to

$$a^{n-k}b^k,$$

corresponding to choosing k of the brackets to be b, and the remaining $n-k$ of the brackets to be a.

The number of terms which simplify to $a^{n-k}b^k$ will equal the number of ways we can pick k brackets out of the n. As explained above, the number of ways to do this is

$$\frac{n(n-1)(n-2)\cdots(n-(k-1))}{k(k-1)(k-2)\cdots 1}.$$

This is because there are n ways to choose the first bracket, $(n-1)$ ways to choose the second bracket, and so on; we stop when we have chosen k brackets, so after k factors. But that would be the number of ways to choose them in order; we have over-counted by the number of ways we could rearrange those k choices. The number of ways to rearrange k things is k choices for what goes first, $(k-1)$ choices for what goes second, and so on, which gives the denominator of the fraction above.

Hence, the expansion of $(a+b)^n$ simplifies to $n+1$ terms, one for each $k = 0, 1, 2, 3, \ldots, n$. The term corresponding to k is

$$\frac{n(n-1)(n-2)\cdots(n-(k-1))}{k(k-1)(k-2)\cdots 1} a^{n-k}b^k.$$

The statement of the binomial theorem given above just lists out the terms for $k=0$, $k=1$, $k=2$, and so forth, up to $k=n$. □

EXERCISE 18: Make sure that you understand the proof of the Binomial Theorem fully. Make sure you can reproduce the reasoning on your own. We will be using this proof again later. (In particular, if you find the counting argument confusing, I would suggest explicitly listing the choices for some examples. Like, maybe try listing all ways of picking 3 brackets from $(a+b)^5$ to obtain a^2b^3. First list all the ways including the order in which you pick the "b" brackets, and then see how many times you have over-counted.)

NOTATION

There is some notation that is helpful.

The product

$$k(k-1)(k-2)\cdots 3\cdot 2\cdot 1$$

happens frequently, so it has a special name ("k factorial") and a special symbol:

$$k! = k(k-1)(k-2)\cdots 3\cdot 2\cdot 1.$$

The binomial coefficients also have a special symbol: if we are expanding $(a+b)^n$, then the coefficient of the kth term, $a^{n-k}b^k$, is written with the symbol

$$\binom{n}{k}.$$

For example, $\binom{4}{2} = 6$. (Remember that we are counting starting with $k=0$, so in the expansion of $(a+b)^4$, the term a^4 is the *zeroth* term, and $4a^3b$ is the *first* term.)

With this notation, the binomial theorem implies that

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k(k-1)(k-2)\cdots 1}.$$

With the factorial notation, we could also write this as

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!}.$$

The binomial coefficient $\binom{n}{k}$ is also said aloud "n choose k", because it is the number of ways of choosing k things out of n things (without regard to order).[2]

Finally, the "summation notation" is sometimes useful. This is like a computer loop instruction: the Greek letter Σ (capital sigma), standing for "sum", means to sum the the thing for a range of values of the "index". For example,

$$\sum_{k=2}^{5} x^k = x^2 + x^3 + x^4 + x^5.$$

The "$k = 2$" tells you the index is k , and it tells you to start the sum with $k = 2$. The value of k increments by one each term, up to the max value of 5.

Using all this notation, we can rewrite the binomial theorem as follows:

BINOMIAL THEOREM (REWRITTEN): Suppose that a and b are any real numbers, and that n is a positive whole number. Then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!}.$$

□

I want to emphasize that all of this section is just *notation*; I have not done anything new at all in this section, except introduce some new symbols.

PASCAL'S TRIANGLE

If you write out just the binomial coefficients for the various powers (omitting the $a^{n-k}b^k$), some appealing patterns start to emerge:

$n = 0$:						1					
$n = 1$:					1		1				
$n = 2$:				1		2		1			
$n = 3$:			1		3		3		1		
$n = 4$:		1		4		6		4		1	
$n = 5$:	1		5		10		10		5		1

This pattern is called "Pascal's Triangle" in most of Europe and the US.[3]

[2] Many books will go further with this notation, and write $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. This is using the $(n-k)!$ on the bottom to cancel the part of the $n!$ on the top you don't want. (Try it for a particular n and k to see how it works!) While this looks neat, I don't want to use it. The reason is that I will want to use the binomial coefficient when n is not a positive whole number. My original formula will still make sense in that case, but this simplified version will not make sense.

[3] Study of this pattern goes back at least to the Indian mathematician Pingala in the 2nd century BCE. It was definitely well known by mathematicians in India and in Persia in the 10th century, and by mathematicians in China in the 11th century. Even in Europe, it was known well before Pascal. The name Pascal got attached to it because he wrote a paper in 1655 which described its properties and which was influential in Europe. However, it seems most or all of the properties he discovered were known hundreds of years before.

Exercise 19: If you don't know this already, see if you can figure out how to calculate each entry in a row from two of the entries in the previous row. (If you get stuck, look it up.) Use this rule to write out more rows of the triangle, up to $n = 8$ (or go further until you get bored!).

Exercise 20: If you want to know why this rule works, here is a hint, for how you obtain e. g. row $n = 5$:

$$\begin{array}{ccccccc} & 1 & 4 & 6 & 4 & 1 & \\ + & & 1 & 4 & 6 & 4 & 1 \\ \hline & 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

Use this hint together with the first way of expanding $(a+b)^5$, that I talked about near the beginning of this assignment.

Conclusion

There is much more to say (perhaps surprisingly!) about the binomial coefficients and the binomial theorem. However, at this point I have talked about everything that we are going to need in this class.

In the next "preamble" assignment, I'll talk about what happens when the n in the binomial theorem becomes *infinite*.

In the class, we will be using the binomial theorem in several places, both what you've done on this assignment, and also through the applications in the next "preamble" assignment.