INTRODUCTION

> Get one percent better every day, and you'll be thirty-seven times better at the end of the year.—(Source unknown)

I couldn't find the source of this quote. The assertion is that improving just incrementally every day can lead to large gains, because the gains accumulate.

Here is the math: if G is how good you are on a particular day, then *adding* one percent to G is the same as *multiplying* G by 1.01:

$$G + (0.01)G = 1G + (0.01)G = G\big(1 + 0.01\big) = G(1.01).$$

So, if G_0 is how good you are on day zero, then to add one percent for 365 days, we multiply by 1.01, a total of 365 times:

$$G_{365} = G_0\big(1 + 0.01\big)^{365} = G_0(1.01)^{365},$$

and

$$(1.01)^{365} \doteq 37.783;$$

you have improved by 3678%!

(Of course, in most parts of life, no amount of training will allow you to improve that much: there is no way you can run 37 times faster than you do now. Practically, even a one percent improvement is not sustainable over time. But the assertion—that a small percentage growth will accumulate dramatically over time—is solid.)

EXERCISE 1: Suppose you could sustain a one percent improvement per *week*, for one year. Find how much you would improve by the end of the year. (Use a calculator or calculator app.)

In this third "preamble" assignment, I will talk about exponential growth and decay. This involves *exponential* functions, $y = a^x$. In particular, when the growth is *continuous*, we get the natural exponential function, $y = e^x$. I will explain how this all works, and explain some properties of natural exponential function.

I will not try to explain how this function is useful in probability; I'll leave that for when the class starts.

Even if the natural exponential function is familiar to you, I will discuss some properties we will need that you may not have seen, so I would recommend looking through this assignment, even if the topic seems well-known to you. If the topic is not familiar to you, please study this assignment carefully; we will be using the natural exponential function a lot.

SIMPLE INTEREST

A standard way of explaining exponential functions is by way of earning interest on money. This goes back to Jacob Bernoulli in 1683. I will follow this procedure, and talk about earning interest on money in the next couple of sections. We will not be using interest in any of our applications, but it seems to be the simplest way of explaining the idea.

Let's start first with *simple interest*. This means that you have an investment, and you earn interest only on the initial investment, not on the interest.

For example, if you invest \$1000, and you earn 10% per time period (say one year), then you will earn \$100 per time period. It will take you 10 time periods to double your investment:

$$\$1000 + \underbrace{\left(\frac{1}{10}\right)\$1000 + \left(\frac{1}{10}\right)\$1000 + \dots \left(\frac{1}{10}\right)\$1000}_{\text{ten times}} = \$2000$$

If you earned 25% simple interest, it would take you 4 time periods to double your money. If you earned 1% simple interest, it would take you 100 time periods to double your money.

In general, if you initially invest P_0 dollars, and you earn $1/n$ of your money in interest per time period, then it will take you n time periods to double your money:

$$\begin{aligned} P_n &= P_0 + \left(\frac{1}{n}\right)P_0 + \left(\frac{1}{n}\right)P_0 + \dots + \left(\frac{1}{n}\right)P_0 \\ &= P_0 + n\left(\frac{1}{n}\right)P_0 \\ &= P_0\left(1 + n\left(\frac{1}{n}\right)\right) = 2P_0 \end{aligned}$$

DISCRETE COMPOUND INTEREST

If the interest is paid to you as you earn it, and then you earn interest on the interest, this will earn more money. This is called *compound* interest.

This is similar to the example that I discussed in the introduction. Suppose you make an initial investment of \$1000, and that you earn 1% interest per month. If you earn interest on your interest, then your *total* amount of money will increase by 1% each month; your increases will accumulate. *Adding* 1% to your money is the same as *multiplying* your balance by 1.01:

$$\$1000 + \$1000(0.01) = \$1000(1) + \$1000(0.01) = \$1000\big(1 + 0.01\big).$$

So if you earn interest for t months, then your final balance P after t months will be[1]

$$P = (\$1000)(1.01)^t.$$

This is called an *exponential* function (because the variable t is in the exponent).

EXERCISE 2: Suppose we invest an initial amount P_0,[2] and we earn an interest rate x per time period (where x is written as a decimal; for example, 1% would mean $x = 0.01$). Find the formula for the amount P of money we would have after t time periods.

EXERCISE 3: Suppose that we instead have an initial investment P_0 that *loses* value at a rate of x per time period. Find the formula for its value P after t time periods.

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[1]For some reason, the amount of money you have is called your "principal" in finance, so the letter P is traditional here.

[2]It is common in math to denote an *initial* value with a subscript 0, meaning the value at time zero.

Comparing Different Compounding Periods

In this section, I want to fix the total time of the investment, and compare the effects of compounding a different number of times, during that fixed total time.

For simplicity, let's take our time period to be the time it would take you to double your money with simple interest. For concreteness, you can imagine that you are earning (somehow!) 100% interest for one year.

With simple interest, it doesn't matter how many payments are made. You could get one payment equal to your investment, at the end of one year; you could get two payments of 50%, or 1/2, your investment, twice (every six months); you could get a payment of 1/12 of your investment each month. In all these cases, your initial investment would be multiplied by 2 after one year.

However, if the interest is compounded, then it matters how often the interest is paid.

If the interest is paid in two payments of 50%, or 1/2, your money, one payment at 6 months and one after 12 months, then you will also earn money on your interest. Say you start with \$1000 dollars; at the end of six months, you will have

$$\$1000 + (0.50)\$1000 = \$1500.$$

Then at the end of twelve months, you will have

$$\$1500 + (0.50)\$1500 = \$2250.$$

Your money has more than doubled in a year, because you have earned 50% interest for six months on the 500 interest from the first six months.

Note as before that *adding* 50% to your total is the same as *multiplying* your total by 1.50:

$$\$1000 + (0.50)\$1000 = \$1000\big(1 + 0.50\big).$$

So, adding 50% again is the same as multiplying by 1.50 again: the total amount you have after twelve months is

$$\$1000\big(1 + 0.50\big)\big(1 + 0.50\big) = \$2250,$$

because you have added 50% twice. If you start with P_0 dollars, at the end of the year you will have

$$P_1 = P_0\big(1 + 0.50\big)^2 = (2.25)P_0$$

dollars.

Let's say instead that you are paid 25%, or 1/4 your total, four times during the year. Adding 25% is the same multiplying by 1.25, or by $1 + 1/4$:

$$P_0 + (0.25)P_0 = P_0 + \left(\frac{1}{4}\right)P_0 = P_0\left(1 + \frac{1}{4}\right).$$

So adding 25% four times (compounding the interest) is the same as multiplying by 1.25 four times over: your final balance will be

$$P_1 = P_0\left(1 + \frac{1}{4}\right)\left(1 + \frac{1}{4}\right)\left(1 + \frac{1}{4}\right)\left(1 + \frac{1}{4}\right) = P_0\left(1 + \frac{1}{4}\right)^4 = (2.44140625)P_0$$

You have earned even more, because you are earning interest on the interest for longer.

Let's say instead that you are paid the interest monthly. Since your interest rate was 100%, this means that you are paid 1/12 of your balance, twelve times during the year. Each time you earn 1/12 interest, your balance is multiplied by $(1+1/12)$. So after a year, your final balance will be

$$P_1 = P_0 \underbrace{\left(1+\frac{1}{12}\right)\cdots\left(1+\frac{1}{12}\right)}_{12 \text{ times}} = P_0\left(1+\frac{1}{12}\right)^{12} \doteq (2.61303529)P_0.$$

If your interest was added daily, your balance would be multiplied by $(1+1/365)$, once each day, for 365 days, and your final balance would be

$$P_1 = P_0\left(1+\frac{1}{365}\right)^{365} \doteq (2.71456748)P_0.$$

If your interest was added every hour, your balance would be multiplied by $(1+1/8760)$ every hour, for all 8760 hours of the year, for a final balance of

$$P_1 = P_0\left(1+\frac{1}{8760}\right)^{8760} \doteq (2.71812669)P_0.$$

If your interest was added every second, your balance would be multiplied by $(1+1/31,536,000)$ every second, for all $31,536,000$ seconds of the year, for a final balance of

$$P_1 = P_0\left(1+\frac{1}{31,536,000}\right)^{31,536,000} \doteq (2.71828178)P_0.$$

CONTINUOUSLY COMPOUNDED INTEREST AND ORGANIC GROWTH

If we continue imagining this process, we will add an interest of $1/n$ to our balance at each step, by multiplying the balance by $(1+1/n)$. We will do so n times, for a final balance of

$$P_1 = P_0\left(1+\frac{1}{n}\right)^{n}.$$

As we let the value of n increase larger and larger, (compounding more and more often), our final balance reaches a limiting value:[3]

$$P_1 = P_0 \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} \doteq 2.718\,281\,828\,459\,045\,235\,360\,287\,471\,352\,662\,497\,757\,247\,093\,699\,95\ldots$$

We think of this limit as *continuous compounding*: the interest is trickling into the account continuously, and interest is continuously being earned on the added interest.

In the same time that simple interest would cause the investment to be doubled, continuous compound interest multiplies the investment by about

$$2.718\,281\,828\ldots$$

This number is called e.

This setup is also important for many examples of *organic growth*. For example, a cell culture, or an organism like a tree, might be modeled as growing at a constant percentage rate. The new growth also grows (like compound interest), and

[3]The symbol "$\lim_{n\to\infty}$" means to find the number that the expression gets closer and closer to, as you take the value of n larger and larger. But you might be worried: do we know there really is such a number? For a more complete explanation, you will have to take Calculus and Analysis! But for the purposes of this course, just the rough idea I've said above is sufficient.

the growth does not happen at sudden discrete intervals, but continuously (like continuous compounding). The new growth is added continuously, and contributes to the growth as it is added.[4]

CONTINUOUS GROWTH WITH OTHER RATES

In the example above, I assumed 100% interest rate per year, for one year. Of course, this is unrealistic. Let's say that the interest rate has a different value, and see how that is calculated.

To start, let's assume that the interest rate is 10% per year, and let's assume we are investing for one year.

If we earn simple interest, then our investment simply grows by 10% total in one year.

Suppose now we earn discretely compounded interest. If the interest is paid, and compounded, a total of n times per year, then at each payment we will earn $(0.10)/n$ of our balance in interest. That is, at each interest payment, our balance will be multiplied by

$$\left(1 + \frac{0.10}{n}\right).$$

We will earn this interest a total of n times in the year, so the initial balance will be multiplied by this factor n times over. That is, at the end of the year, our final balance will be

$$P_1 = P_0 \left(1 + \frac{0.10}{n}\right)^n.$$

For example, if the interest is compounded monthly, then at the end of the year our balance will be multiplied by

$$\left(1 + \frac{0.10}{12}\right)^{12} \doteq 1.104713,$$

or an effective rate of 10.4713%.

If the interest is compounded continuously, then at the end of the year our balance will be

$$P_1 = P_0 \lim_{n\to\infty} \left(1 + \frac{0.10}{n}\right)^n \doteq (1.105170918)P_0,$$

or an effective rate of 10.5170918%.

We can actually do this computation in terms of the number e. Let's do it for any interest rate, not just for 10%.

Suppose our interest rate is x per year (expressed as a decimal). Suppose that the interest is compounded n times a year. Each time we add interest, we multiply our balance by

$$\left(1 + \frac{x}{n}\right).$$

After the end of one year, we have added interest n times, so we have multiplied by this factor n times. So, at the end of the year, the balance is

$$P_1 = P_0 \left(1 + \frac{x}{n}\right)^n.$$

[4] Of course, a cell culture or a tree is made up of a discrete set of cells, which do reproduce in discrete steps; but the number of cells is so enormous, that continuous growth is a simpler and more accurate mathematical model than discrete growth.

If we now continuously compound, then after one year the balance is

$$P_1 = P_0 \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

How can we calculate this number? Here is a trick: define a new variable m, by

$$m = \frac{n}{x}.$$

Then we have that

$$\frac{1}{m} = \frac{x}{n}, \qquad \text{and} \qquad n = mx.$$

(Check my algebra!)

Also, when n gets really large, so does m. Substituting these into our expression above, we find that

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{mx}.$$

(Again, check my algebra!)

Now, we can additionally use a trick of applying an exponent rule, that $a^{bc} = (a^b)^c$, so we get:

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{mx} = \left(\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m\right)^x.$$

Now, check out the bracket in the last step:

$$\left(\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m\right) = e.$$

So we finally get

$$P_1 = P_0 \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = P_0 \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{mx} = P_0 \left(\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m\right)^x = P_0 e^x.$$

Summarizing:

$$\boxed{\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x}$$

EXERCISE 4: This isn't exactly an exercise, but I strongly suggest that you memorize the formula in the box above. It is going to come back several times later. It would also be valuable to remember how it was derived, if you are feeling ambitious.

For our practical compound interest example, we have showed the following:

> If you invest P_0 dollars, and earn an interest of x per year (expressed as a decimal), compounded continuously, then at the end of the year, your balance will be
>
> $$P_1 = P_0 e^x.$$

For example, if the interest rate is 10% per year, then after one year of continuous compounding, the balance is multiplied by the factor

$$e^{0.10} \doteq 1.105170918,$$

or an effective rate of 10.5170918%, the same answer as we got before.[5]

[5] You can calculate this with the e^x button on your calculator or calculator app.

EXERCISE 5: Suppose that we have an initial investment (or other quantity) P_0, that *loses* value at a rate of x per year, compounded continuously. Repeat the whole derivation given above for this case. Your final result should be a formula for P_1, the amount remaining after one year. (Your formula should look similar to $P_1 = P_0 e^x$, with a small change. You might be able to guess it, but I still suggest going through the whole derivation to figure it out for certain. This will also give you practice understanding the derivation of the formula, which we will need again.)

CONTINOUS GROWTH FOR OTHER TOTAL TIME PERIODS

So far, I have been assuming continuous compounding for a fixed total period of one year. (I changed the time period that we were adding and compounding interest, which changed the number of times interest was added during the one year. I eventually let the number of compounding periods become infinitely big. But I still kept the total time period as one year.)

Let's suppose we had a different time interval. For example, let's say that we had 10% interest, compounded n times a year, but that we invested for 5 years. Then each addition of interest would multiply the balance by

$$\left(1 + \frac{0.10}{n}\right)$$

as before. However, we would add the interest n times a year for 5 years, for a total of $5n$ times. So after 5 years, our balance would be

$$P_5 = P_0 \left(1 + \frac{0.10}{n}\right)^{5n}.$$

If we take the limit of continuous compounding, we would multiply our original investment by

$$P_5 = P_0 \lim_{n\to\infty} \left(1 + \frac{0.10}{n}\right)^{5n} \doteq 1.6487,$$

or an increase of about 64.87% (compare simple interest which would increase the investment by 50%).

More generally, if we have an interest rate of x per year (expressed as a decimal), we are compounding n times a year, and we invest for t years, then after t years, our balance will be

$$P_t = P_0 \left(1 + \frac{x}{n}\right)^{nt}.$$

If we are compounding continuously, we take the limiting value as n gets large. To write this in terms of e, we can make a substitution $m = nx$ like before:

$$P_t = P_0 \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{nt} = P_0 \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{mxt} = P_0 \left(\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{m}\right)^{xt} = P_0 e^{xt}.$$

Summarizing:

$$\boxed{\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{nt} = e^{xt}}$$

For our practical example:

If you invest P_0 dollars, and earn an interest of x per year (expressed as a decimal), compounded continuously, then at the end of t years, your balance will be

$$P_t = P_0 e^{xt}.$$

Said more generally:

If a quantity has an initial value of P_0, and grows continuously at a rate x per time period (expressed as a decimal), then at the end of t time periods, the quantity will be

$$P_t = P_0 e^{xt}.$$

EXERCISE 6: As in the last exercise, find the formula for the case where the quantity *decreases* continuously at a rate of x per time period. You can probably guess it by now, but go through the whole derivation to be sure. (We will use the derivation again later, not just the result.)

EXERCISE 7: Suppose that you invest an initial amount of \$1000, that you earn 10% interest per year, and that you invest for 5 years total. Compare your total balance, at the end of five years, in the case where:
(a) your interest is simple (not compounded)
(b) your interest is added and compounded yearly
(c) your interest is added and compounded monthly
(d) your interest is added and compounded continuously.
(Use a calculator or calculator app to do the computations.)

THE NATURAL LOGARITHM

Sometimes it is necessary to solve an equation involving an exponential function. For this, we often need a way to invert the exponential function: this is the logarithm.

Here is the rule which defines a logarithm to any base:[6]

$$\text{for any } x, \qquad \log_a (a^x) = x \qquad \text{and} \qquad a^{\log_a x} = x.$$

We will only ever need logarithms for the natural exponential function, where $a = e$. In this case, the logarithm has a special symbol: $\log_e y$ is written $\ln y$. That is:

$$\boxed{\text{for any } x, \qquad \ln (e^x) = x \qquad \text{and} \qquad e^{\ln x} = x.}$$

Here is an example of how it is used. Suppose that we have an initial investment of \$1000, earning 100% interest per year, compounded continuously. By what we figured out above, after t years, our balance would be

$$P_t = (\$1000)e^t.$$

Now, suppose we want to know how long our money will take to double. That is, we want to find the time t such that $P_t = \$2000$. We therefore need to solve

$$\$2000 = (\$1000)e^t, \qquad \text{that is,} \qquad e^t = 2$$

[6] There are many other ways of thinking of logarithms, and many other sorts of applications for them. In this class, we will only be using them as inverses of exponential functions, so that's the explanation I use here.

for the unknown time t. Apply ln to both sides of the equation:

$$\ln\left(e^t\right) = \ln(2)$$

and apply the rule, $\ln\left(e^{\text{"whatever"}}\right) = \text{"whatever"}$, to get $\ln\left(e^t\right) = t$, so

$$t = \ln(2) \doteq 0.693.$$

Therefore it takes about 0.693 years, or about 8.3 months, for the money to double. (You can use a calculator or calculator app to find ln(2).)

EXERCISE 8: Suppose that we invest \$1000, and that we earn 5% interest per year, compounded continuously. How long will it take for our balance to reach \$3000? (Note that you can check your answer by substituting it back into your beginning formula.)

CONCLUSION

I have used the examples of growth and decay, particularly compound interest, to explain the natural exponential function. That is not exactly how we will use it in probability, but I think it is still the simplest way to explain the function if you are not familiar with it.

If you are familiar with the natural exponential function, then much of this assignment may have been review for you. I think the thing that was most likely to be new was the infinite product formula:

$$\boxed{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x}$$

and also the version you worked out for *decay* instead of growth:

$$\boxed{\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}}$$

This formula will be particularly important for us. Be sure to remember it, and ideally understand where it comes from.

In Preamble Assignment #4, I will explain how to combine these formulas with the binomial theorem that I talked about in Preamble Assignment #2.