Differentials and derivatives

Recall the example I walked through in class: I started with the relationship

$$y = x^2.$$

Then, I wanted to let x change to $x + \Delta x$, which would make y change to $y + \Delta y$. I wanted to find how the change in the output, Δy, depended on the change Δx in the input, and I also wanted to find the ratio of changes, $\frac{\Delta y}{\Delta x}$.

Then, I wanted to see how this all simplified if the change in x was *differential* or "infinitely small": I assume x changes differentially to $x+\mathrm{d}x$, y changes differentially to $y + \mathrm{d}y$. I wanted to then find how the differential change in the output, $\mathrm{d}y$, depends on the differential change in the input, $\mathrm{d}x$, and I wanted to find the ratio of changes, $\frac{\mathrm{d}y}{\mathrm{d}x}$, also called the *derivative*.

Since the differential changes are my main goal, I'll start from the beginning with $\mathrm{d}x$ and $\mathrm{d}y$ (rather than Δx and Δy as in the notes).

So, if x and y are the starting values, I then let the

$$x \text{ increase to } x + \mathrm{d}x$$

and that causes the

$$y \text{ to increase to } y + \mathrm{d}y.$$

Since the two variables are still related , I have the relation

$$y + \mathrm{d}y = (x + \mathrm{d}x)^2.$$

Simplifying, I find

$$y + \mathrm{d}y = x^2 + 2x\,\mathrm{d}x + (\mathrm{d}x)^2.$$

Since $y = x^2$, I can subtract the y from the left side and the x^2 from the right side (subtracting equal things from each side!). So I get

$$\mathrm{d}y = 2x\,\mathrm{d}x + (\mathrm{d}x)^2.$$

This is valid whatever the change $\mathrm{d}x$ in x is, even if it is big. But I want the change $\mathrm{d}x$ to be small—in fact, infinitely small. If $\mathrm{d}x$ is small—a tiny bit of x—then $(\mathrm{d}x)^2$ will be that much smaller still—a tiny bit of a tiny bit. In the limit, where $\mathrm{d}x$ is "infinitely small", we imagine that $(\mathrm{d}x)^2$ is negligible altogether. So we introduce the odd-looking rule:[1]

$$\boxed{(\mathrm{d}x)^2 = 0}$$

Putting this into the previous equation, we find

$$\mathrm{d}y = 2x\,\mathrm{d}x.$$

This is the *relationship of the differentials*.

Dividing both sides by $\mathrm{d}x$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x.$$

This ratio of differential changes is called the *derivative*.

I will talk more about the graphical interpretation later, but it might be worth mentioning now, since it may help you understand what is going on. Imagine the graph of the relationship

$$y = x^2,$$

[1] In more formal terms, we are looking only at the *linear* approximation to the relationship of $\mathrm{d}y$ and $\mathrm{d}x$.

(please draw it!), and imagine that (x, y) is some point on the graph. Then when we increase x a little bit, saying that the relationship still holds is saying that the shifted point

$$(x + \mathrm{d}x, y + \mathrm{d}y)$$

is also on the graph. The value of $\mathrm{d}x$ is the "run" between these two points, and the value of $\mathrm{d}y$ is the "rise". Dividing $\mathrm{d}y$ by $\mathrm{d}x$ allows us to find the slope:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

I needed to take two points that were very close together, so that the slope of the graph did not have a chance to change. For example, at the point $(2, 4)$, we know $x = 2$, so the slope of the graph at that point is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x = 2(2) = 4.$$

What does "slope" of a curved graph actually mean? If we zoomed in on the graph at that point, it would look like a straight line, which goes up 4 units for every 1 unit it goes over. We have to zoom in enough that it looks like a straight line, so that there is one definite slope; if we don't zoom in enough, the slope is changing, and not well-defined.[2]

Eventually, our goal will be to be able to do this with *any* given relationship. Given any relationship between two variables, we will want to be able to figure out the relationship between their differentials.

Exercises on finding relationships of differentials

1. For each of the relationships given, repeat the process above, to find the relationship of the differentials $\mathrm{d}y$ and $\mathrm{d}x$, and to find the ratio (derivative) $\frac{\mathrm{d}y}{\mathrm{d}x}$. Simplify your answers as much as possible.[3]
   (a) $y = x^2$ (Do it yourself again without notes to make sure you followed everything.)
   (b) $y = x^3$ (I briefly did this in class but left out details; do it yourself again to make sure you get it. You can shorten the calculations by using $(\mathrm{d}x)^2 = 0$ as you go.)
   (c) $y = x^4$ (I would suggest canceling $(\mathrm{d}x)^2$ terms as you go, and/or using things you already worked out, to save work here!)
   (d) $y = x^5$ (Same comment!)
   (e) Do you see the pattern for $y = x^n$, where n is a positive whole number? Can you explain (similar to what you did above) why this pattern should work for any positive whole number n?
   (f) $y = x$. (Does this fit with the rule you found for $y = x^n$ in the previous part?)
   (g) $y = 1$ (Does this fit with the rule you found for $y = x^n$ in the previous part? It might be a little confusing; just follow the rules consistently...)

[2] Of course, the graph is always a little bit curved, no matter how far we zoom in. That is why we must take $\mathrm{d}x$ *infinitely small*, not just really small.

[3] If you have taken calculus before, you may see some shortcuts to the answer; I would like you to do the algebra anyway, just to see how it works out.

2. For each of the relationships given, find the relationship of the differentials $\mathrm{d}y$ and $\mathrm{d}x$, and find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Simplify your answers as much as possible.[4]
   (a) $y = 5x^2$
   (b) $y = ax^2$, where a is any constant.
   (c) $y = x^2 + 3$
   (d) $y = x^2 + b$, where b is any constant.
   (e) $y = ax^2 + bx + c$, where a, b, and c are any constants.

[4]Same note as previous question!