More on differentials and derivatives

In these problems, I will continue what we were doing with the previous problem set. I will assume that we have some relationship between variables, and I will be looking for the relationship between their differential changes. All the examples we had before were y a polynomial function of x; I will try more complicated relationships this time.

1. For each of the relationships given, find the relationship of the differentials $\mathrm{d}y$ and $\mathrm{d}x$, and find the ratio (derivative) $\frac{\mathrm{d}y}{\mathrm{d}x}$. Simplify your answers as much as possible. If your answer for $\mathrm{d}y$ depends on y, rewrite it so that it depends on x only. Some of these might be tricky; think about them, do what you can, and we'll talk about them in class next time![1]
   (a) $y = \sqrt{x}$. (Note that $y = \sqrt{x}$ is, by definition, a number with the property that $y^2 = x$.)
   (b) $y = \sqrt[3]{x}$. (Note that $y = \sqrt[3]{x}$ is, by definition, a number with the property that $y^3 = x$.)
   (c) $y = \sqrt{x^3}$.
   (d) $y = \dfrac{1}{x}$.
   (e) $y = \dfrac{1}{x^2}$.
   (f) $y = \dfrac{1}{\sqrt{x}}$.
2. (a) The area of a circle with radius r is given by the formula
   $$A = \pi r^2.$$
   Suppose that the radius is increased differentially, from r to $r + \mathrm{d}r$. What is the corresponding increase in the area $\mathrm{d}A$? What is the derivative $\dfrac{\mathrm{d}A}{\mathrm{d}r}$?
   (b) Can you use a diagram, showing the differential change in radius and area, to explain why the formula you got above is what it is?
   (c) The volume of a circular cylinder, with radius r and height h, is given by
   $$V = \pi r^2 h.$$
      (i) Suppose that the radius increases from r to $r + \mathrm{d}r$, while the height stays constant. What is the corresponding change in volume $\mathrm{d}V$, and derivative $\dfrac{\mathrm{d}V}{\mathrm{d}r}$?
      (ii) On the other hand, suppose that the height increases from h to $h+\mathrm{d}h$, while the radius stays constant. What is the corresponding change in volume $\mathrm{d}V$, and derivative $\dfrac{\mathrm{d}V}{\mathrm{d}h}$?
      (iii) For each of those two previous cases, can you explain the formula you got for $\mathrm{d}V$ by drawing a picture of the differential change?
      (iv) Suppose that both r and h change simultaneously. Find a relationship between $\mathrm{d}V$, $\mathrm{d}r$, and $\mathrm{d}h$.[2] Can you explain this formula with a diagram as well?

[1] Even if you see a pattern that would let you jump to the final answer, please do the actual computations. In particular, if you have taken calculus before, you may see some shortcuts to the answer; I would like you to do the algebra anyway, just to see how it works out.

[2] It doesn't make sense to write a derivative the same way as before, since three things are changing. We'll see a way of writing this in terms of derivatives soon, though.