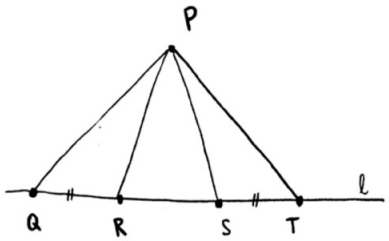
NEWTON'S EXPLANATION OF KEPLER'S 2ND LAW, AND AREAS

These problems concern Newton's explanation of Kepler's 2nd law, which I explained in class (see the video and notes for Class #4, Monday March 1 for details).

Note that some of these questions are challenging and/or open-ended. They require you to think about them more creatively, and you may very well get stuck or have only partial progress. Comments on this:

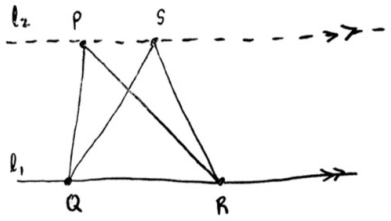
- It is OK to be stuck. Try different things. Draw pictures. Use wishful thinking: try a special case which is easier to solve, and then, if you can solve it, see if you can use that to get back to the general case. It may take you a while (days, weeks) to solve some of these.
- You don't have to do the problems in order! Some hard problems are followed by easy problems.
- Don't worry about totally solving everything necessarily. If you are spending 3 hours between classes, that is about the right effort.
- PLEASE DO NOT LOOK UP OUTSIDE SOURCES. I'm trying to get you to develop a certain kind of thinking here. Looking up methods or answers will not help you do so.
- Similarly, I wouldn't try too hard to recall solutions or methods you might have see before; it will be more constructive to work out the idea again anyway.
- I promise that everything I ask is solvable with what you know already and what we have done in class! If you are stuck, it is NOT because you don't know something.

1. In the explanation of Kepler's second law, two key steps are facts about areas of triangles. These facts are proven in Euclid, and you may have seen them in other classes before. I would like you to try to prove these *without* looking things up. (When I say "prove", I'm not looking for a very formal proof, like you might have done in geometry in high school. On the other hand, I am looking for a pretty solid logical argument.)
   (a) The first property is that, if two triangles (i) have bases QR and ST which are on the same line ℓ, (ii) have bases QR and ST the same length, and (iii) have the same third point P, then the triangles PQR and PST have the same area. Why is this true? (As I mentioned above, you might start trying to prove special cases, or related easier facts. Note that when I say "prove", what I'm asking is I want to understand *why* this is true. Please don't look it up—try to think it through yourself!)[1]



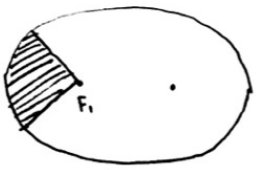
[1] You might be tempted to say, since $A = \frac{1}{2}bh$, and the triangles have the same b and h, then the areas are the same. This isn't the way I'm thinking of proving this. However, if you do want to argue that way, then my question becomes: why is the formula $A = \frac{1}{2}bh$ true for all triangles?

(b) The second property is that, if two triangles (i) share the same base QR sitting on line ℓ_1, and (ii) have their third points P and S lying on a line ℓ_2 which is parallel to ℓ_1, then the triangles PQR and SQR have the same area. Why is this true? (The same comments apply as before.)

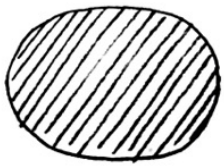


(c) (To be done after you have at least a partial solution to the previous two parts) It is also possible to give a proof of the previous two properties using a calculus kind of idea! I'll just give a hint right now: think of slicing things into very (infinitely) thin pieces...

2. In class, I went through Newton's explanation of Kepler's 2nd law (see the video and notes for Class #4, Monday March 1 for details). I'd like you to understand and be able to reproduce this argument. Since we do not have tests in this class, I will ask it as a (sort of unusual) homework problem: recreate the proof of Kepler's 2nd law for yourself, referring to your notes as little as possible. Try to identify the points in the argument where you get lost, and try to identify the assumptions that are made in the argument.
3. Kepler's 1st law says that the planets travel on ellipses, with the sun at one focus. This means that Kepler's 2nd law concerns areas of segments of an ellipse, created by two lines coming from the focus:



This raises the question: how do I find the area of such a segment? This is a tricky question, so I'd like to start with an easier question: how do I find the area of an ellipse?



I'll have more to say about ellipses later, but here's what you need to know now: the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are some non-zero constants. (From this you should be able to find the x and y intercepts, and see roughly what the shape is.) Based on this, try to find the formula for the area of an ellipse. Comments:

- This will involve the sort of calculus ideas that we have been talking about. However, it does NOT require you to know anything you have not yet done.
- Partial results are a good way to start! In particular, you could look for estimates: the area is definitely bigger than (blank) and less than (blank) would be helpful.
- Try various sorts of wishful thinking.
- This is tough, if you're stuck, go on to other problems and come back to this one.

More on differentials

4. (Optional) In Problem Set #1, you saw the pattern that

$$\text{if} \qquad y = x^n \qquad \text{then} \qquad \mathrm{d}y = nx^{n-1}\mathrm{d}x,$$

for any whole positive number n. Some people asked about the proof. One way to prove this is by the "binomial theorem". If you know this theorem, you can prove it that way. I want to discuss the binomial theorem a little later (we're going to use it for other things too, and generalize it), but I don't want to get into it yet. Here's another way of proving this, without needing the binomial theorem. You do need to understand proof by induction; if you don't know it, and you're interested, I recommend looking up *Book of Proof* by Richard Hammack, which is available free online (induction is Chapter 10).

(a) Prove the formula

$$(x + \mathrm{d}x)^n = x^n + nx^{n-1}\mathrm{d}x \tag{1}$$

for all whole positive numbers n by induction. More explicitly, here are the steps:

(i) Prove that the formula (1) is true when $n = 1$.

(ii) *Assume* that the formula (1) is true for *some* positive integer value of n. Based on that assumption, *prove* that the formula must *also* be true when n is replaced with $n + 1$.

By the principle of mathematical induction, those steps prove that the formula (1) is true for all positive integer n. (See Hammack for more details if that is confusing.)

(b) Now prove the main claim based on the previous part (this part is easy).

5. (a) (i) Suppose that A is the area of a circle of radius r, and C is the circumference. Without using the formulas for A and C, yse a diagram to argue that we must have

$$\frac{\mathrm{d}A}{\mathrm{d}r} = C.$$

(ii) Check that this agrees with the formulas $A = \pi r^2$ and $C = 2\pi r$.

(iii) Suppose that the radius, circumference, and area of a circle are all changing with time. Then, for example, $\frac{\mathrm{d}A}{\mathrm{d}t}$ represents how quickly the area is changing with respect to time. (Think this through!) Find a formula for rate of change of area, in terms of C and $\frac{\mathrm{d}r}{\mathrm{d}t}$.

(b) (i) Suppose that V is the volume of a sphere of radius r. Without using the formula for V, (I'm not assuming you know it!), can you explain what $\frac{\mathrm{d}V}{\mathrm{d}r}$ is geometrically?

(ii) The volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. We are going to see where this odd-looking formula comes from very soon! For the moment, I would like you to use this formula to find the formula for the surface area S of a sphere of radius r.

(c) (Challenging) You can model a growing raindrop as a sphere, whose radius is increasing with time. The raindrop accumulates water molecules from the air, which causes it to grow.

(i) Use these assumptions to make an equation which relates the rate of growth in volume, $\frac{\mathrm{d}V}{\mathrm{d}t}$, to the other properties of the sphere. (This is called a *differential equation*, because it relates the differentials.)

(ii) Do you think that the radius of the raindrop is going to grow more quickly over time, in an accelerating way? Or will the radius grow more slowly as it gets bigger? Or will the radius grow at a constant rate? Just make a guess at this point, and then move to the next part:

(iii) I claim that you can use the differential equation, plus what you worked out in the previous part, to describe exactly how the radius of the raindrop grows over time. Try to do this!