Differential equations: Newton's law

I am going to start the discussion of differential equations with physics, and Newton's law. Differential equations are much more broadly applicable, but I want to start here, partly because this is where they were first invented. Don't worry if you don't know any physics; I am including everything you will need.

Suppose that we have some object, which is free to move in one dimension. (Ultimately, I am interested in motion in two or three dimensions, like the path of a planet in space, or the path of a ball thrown in the air. However, I want to start with motion just along one line, to keep things simpler.) You could imagine a car on a road, or a rock that you are throwing straight vertically into the air. Represent its position by x; the magnitude of x is how far the object is from the point you have taken to be an origin, $x = 0$, and the sign of x indicates the direction. The position x is going to depend on the time t. The *velocity* of the object is defined to be

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}.$$

Note that the velocity v also depends on the time t.

Now, suppose that the object experiences some force F. The force F could depend on the time t, the position x, and/or the velocity v. Now, Newton's law says that the function v satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{F}{m}, \tag{1}$$

where m is the mass of the object.

1. (Constant force) Suppose that the object is subject to a constant force $F = F_0$. That means that the acceleration $a = F_0/m = a_0$ is also a constant.[1] Therefore, we have the following differential equation for v:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a_0. \tag{2}$$

   The meaning of this equation is that v is an unknown function of time t, whose derivative is the constant a_0. To *solve* this differential equation is to find all possible functions v of t that satisfy the equation (2).

   (a) To make things more concrete, let's imagine a_0 is a specific known number, let's say $a_0 = 10$.

      (i) Find a solution to the differential equation (2) in this case, which now says $\frac{\mathrm{d}v}{\mathrm{d}t} = 10$. That is, find a function v of t which satisfies this condition.

      (ii) Is the function v you found the only possible answer? If not, find the others. If so, explain why.

      (iii) Suppose now that we also know the velocity is equal to $v_0 = 20$ when time $t = 0$. Can you find a solution of $\frac{\mathrm{d}v}{\mathrm{d}t} = 10$ which satisfies this further condition? (The differential equation with this extra condition is called an *initial value problem*.)

[1] Elementary physics textbooks often just write "a" for the constant acceleration in this situation, and in the following formulas. This can be confusing, because these formulas will *only* be valid when a is constant, which it usually is not.

(b) (i) Now, assume that a_0 is a known constant, but we do not want to give it a specific numerical value, so that we can get a general formula. Find all possible solutions of the differential equation (2). Your answer will be a formula for v as a function of t, and your formula will involve the time t and the unknown constant a_0.

(ii) Suppose now that we also know that the velocity is equal to some known initial value v_0 at time $t = 0$. Find a solution of (2) which also satisfies this extra condition.

(c) Now, substitute your solution for v as a function of t, (the one you found in the previous part)into the definition of velocity:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v$$

This gives you a differential equation for the unknown position x as a function of time. Give all possible solutions for this differential equation. Your answer will give the position x of the object, as a function of time t, and it will also involve the constant a_0 and the initial value v_0. Finally, assume now that you also know the initial position x_0 of the object at time $t = 0$; find a solution of this differential equation which satisfies this extra condition.

2. (Projectile near earth without air resistance) If an object is dropped, or thrown up in the air, it experiences the force of gravity. If we are near the surface of the earth (not going up too high), and if we take "up" to be positive, then this gravitational force is

$$F = -mg,$$

where m is the mass of the object, and g is a constant, called the acceleration of gravity. (The numerical value of g is approximately 10 meters per second, per second.) If we ignore the resistance of the air, then this is the only force on the object, so we have a special case of the previous question, with $a_0 = -g$, the velocity v being vertical velocity, and the position x being the height.

(a) Suppose that we drop a rock off of a tall building, so $x_0 = 80$ meters (that is about twenty storeys). Based on the equations you found in the previous question:

(i) Make a rough sketch of the graph of velocity versus time.

(ii) Make a rough sketch of the graph of height versus time.

(iii) How long does the rock take to hit the ground?

(iv) How fast is the rock going when it hits the ground? (Your answer will be in meters per second; out of curiosity, use a calculator to translate to miles per hour. Note that leaving out air resistance is quite unrealistic in this situation; we will see how to incorporate it later.)

(b) Suppose that we are again standing on the tall building, $x_0 = 80$ meters, but we throw the rock straight up as hard as we can, $v_0 = 20$ meters per second. (We throw it slightly off the edge, so that it misses the building on the way back down.)

(i) Make a rough sketch of the graph of velocity versus time.

(ii) Make a rough sketch of the graph of height versus time. (Use the same scale for time on both graphs, so you can compare features on one graph to the other.)

(iii) How long does the rock take to reach its maximum height?
(iv) How high does the rock go?
(v) How long does the rock take to hit the ground?
(vi) How fast is the rock going when it hits the ground?

3. (Steadily increasing acceleration) Suppose that we are in a car, and we are gradually depressing the accelerator pedal, in a steady way. We could model this as the force not being constant, but increasing linearly. This means (by Newton's law) that the acceleration is increasing linearly:

$$a = a_0 + jt, \tag{3}$$

where a_0 and j are some constants. (Of course, this might model lots of other situations as well.)[2] Then Newton's law says

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a_0 + kt. \tag{4}$$

Suppose that the initial velocity v_0 and the initial position x_0 are known.

(a) Find the solution of the differential equation (4) which also satisfies $v = v_0$ when $t = 0$.
(b) Use the formula you found for v in the definition of velocity to give a differential equation for x, and find the solution of this differential equation which also satisfies $x = x_0$ when $t = 0$.
(c) Draw a rough sketch of the acceleration, velocity, and position versus time.

[2] In physics, if the acceleration is changing, the rate at which the acceleration is changing is called the "jerk", $j = \frac{\mathrm{d}a}{\mathrm{d}t}$. So this question is about motion with changing acceleration, but constant jerk!