Fractional and negative exponents

Initially, when we define exponents x^n, we define them for positive whole numbers n. For example,

$$x^5 = x \cdot x \cdot x \cdot x \cdot x,$$

by definition. You can see that these exponents satisfy the rule

$$\boxed{x^a \cdot x^b = x^{a+b}} \tag{1}$$

for all whole numbers a and b. For example,

$$x^2 \cdot x^3 = (x \cdot x) \cdot (x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x = x^5.$$

We also define

$$\boxed{x^1 = x} \tag{2}$$

It turns out that the property (1) is important for a number of reasons. We therefore *define* negative and fractional exponents to be consistent with this rule. That is, we use (1) to determine values of x^n when n is not a positive whole number. I will ask you to show how this works out in the problems below.

1. (a) Show that rules (2) and (1) imply that $x^2 = x \cdot x$.
   (b) Similarly, prove that $x^3 = x \cdot x \cdot x$, using (2), (1), and the previous question.
   (c) For any positive whole number n, prove that $x^n = x^{n-1} \cdot x$. (This is called a *recursive* definition.)
   (d) Use (1) (and maybe (2), plus anything you've worked out so far) to find the value of x^0. (The idea is to use only (1), (2), and things you have already worked out. Do *not* assume rules for negative exponents: that is what we are trying to prove! However, if you want, you could solve the next part first, to find x^{-1} from the rules, then you could come back to use that to solve for x^0.)
   (e) Use (1) (and maybe (2), plus anything you've worked out so far) to find the value of x^{-1}. (If you decide to do this part before the previous one, and then to use this answer to find x^0, then you must *not* assume the value of x^0, or you will get trapped in a logical circle!)
   (f) Find x^{-2}.
   (g) Prove that $x^{\frac{1}{2}} = \sqrt{x}$. (Note that, by definition, $\sqrt{x}$ is a positive number such that $(\sqrt{x})(\sqrt{x}) = x$.)
   (h) Prove that $x^{\frac{3}{2}} = \sqrt{x}^3$.
   (i) Prove that $x^{\frac{3}{2}} = \sqrt{x^3}$.
   (j) From the previous two items, conclude that $\sqrt{x}^3 = \sqrt{x^3}$.
   (k) Prove that $x^{\frac{1}{3}} = \sqrt[3]{x}$. (Note that, by definition, $\sqrt[3]{x}$ is a positive number such that $(\sqrt[3]{x})(\sqrt[3]{x})(\sqrt[3]{x}) = x$.)
   (l) Find $x^{-\frac{1}{2}}$.
   (m) Prove that, for any number a, $x^{-a} = (x^a)^{-1} = \dfrac{1}{x^a}$.
   (n) (Optional) Prove that $x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$, for any positive whole numbers m and n.
   (o) (Optional) Prove that $x^{-\frac{m}{n}} = \dfrac{1}{\sqrt[n]{x^m}} = \dfrac{1}{(\sqrt[n]{x})^m}$, for any positive whole numbers m and n.

2. (Optional) The purpose of this question is to justify the rule

$$\boxed{(x^a)^b = x^{ab}} \tag{3}$$

We will justify it for gradually more types of numbers a and b.

(a) Prove that (3) is true when a is any number, and $b = 1$. (Use (2).)
(b) Prove that (3) is true when a is any number, and b is any whole positive number. (Use (1).)
(c) Prove that (3) is true when a is any number, and $b = 0$.
(d) Prove that (3) is true when a is any number, and $b = -1$.
(e) Prove that (3) is true when a is any number, and b is any negative whole number.
(f) Prove that (3) is true when a is any number, and $b = \frac{1}{2}$.
(g) Prove that (3) is true when a is any number, and $b = \frac{m}{n}$, where m and n are any positive whole numbers.
(h) Prove that (3) is true when a is any number, and $b = -\frac{m}{n}$, where m and n are any positive whole numbers.

Fractional exponents and derivatives

3. On Problem Set 1, #1(e), you found the rule

$$(x + \mathrm{d}x)^n = x^n + nx^{n-1}\,\mathrm{d}x, \quad \text{so that if} \quad y = x^n, \text{ then } \mathrm{d}y = nx^{n-1}\,\mathrm{d}x. \tag{4}$$

This was valid when n was a positive whole number.

(a) For each part of Problem Set 2, question #1, rewrite the question in terms of a fractional or negative exponent, and rewrite your answer in terms of a fractional or negative exponent. Does the rule (4) work every time?
(b) Let's try to justify rule (4) when n is not necessarily a positive whole number. First, let's imagine that n is positive but fractional. Suppose $y = x^{\frac{k}{m}}$, where k and m are some positive whole numbers. Find the differential $\mathrm{d}y$ and the derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Simplify as much as possible. (For this problem, assume that the rule (4) is valid for n being any positive whole number.) Then check that your final answer agrees with (4), when you set $n = \frac{k}{m}$.
(c) Now let's suppose that n is a negative fraction. Suppose $y = x^{-\frac{k}{m}}$, where k and m are some positive whole numbers. Find the differential $\mathrm{d}y$ and the derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Simplify as much as possible. (For this problem, assume that the rule (4) is valid for n being any positive whole number.) Then check that your final answer agrees with (4), when you set $n = -\frac{k}{m}$.
(d) (Open-ended) How would you interpret something like $2^{\sqrt{2}}$? If you enter this in a calculator, it will give you a number, but what does this mean? What is the calculator actually finding?

Infinite algebra

These problems concern what I'll call "infinite polynomials". An example would be the expression

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

The "$\cdots$" are supposed to mean that this goes on forever: this expression is imagined to have an x^6 term, an x^{100} term, an x^{21345} term,... Other examples would be

$$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

or

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$$

In these examples, we assume that the coefficients continue with the same pattern. (For this notation to make sense, the pattern of the coefficients has to be obvious; if it isn't obvious, we will have to specify the rule.)

These types of expressions turn out to be surprisingly useful and interesting. I want to start with some examples of doing algebraic manipulations with these types of infinite expressions.

4. Multiply out and simplify the following!
   (a)
   $$(1 - x)(1 + x + x^2 + x^3 + x^4 + x^5 + \cdots)$$
   (b)
   $$(1 + x)(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots)$$
   (c)
   $$(1 + x + x^2 + x^3 + x^4 + x^5 + \cdots)(1 + x + x^2 + x^3 + x^4 + x^5 + \cdots)$$
   (d) Make up something similar and try it out!
5. (a) Use what you calculated above to find a formula for
   $$\frac{1}{1 - x}$$
   as an "infinite polynomial".
   (b) Similarly, find a formula for
   $$\frac{1}{1 + x}.$$
   (c) In light of those things, what can you conclude from the third calculation that you made above, in question 4c?
   (d) Find an "infinite polynomial" formula for
   $$\frac{1}{1 - x^2}.$$
6. What do these crazy calculations mean?
   (a) Try substituting in $x = \frac{1}{2}$ into both sides of your formula for $\frac{1}{1-x}$ that you found in question 5a. Try to figure out what each side is numerically—do they come out equal?
   (b) Try substituting in $x = \frac{1}{2}$ into both sides of each of the other formulas you found in the different parts of question 5, and again check numerically; is it still working out?
   (c) Try the same thing, but with $x = \frac{1}{3}$.
   (d) Same thing, but with $x = 0.1$.
   (e) Now try $x = 2$ in both sides of the formula you found in question 5a. Does it work out numerically now?
   (f) (Open-ended) Any thoughts as to when these formulas make sense, and why?

7. In class, talking about Problem Set 2, the question came up: can we somehow "expand", or "multiply out", something like
$$(x + \mathrm{d}x)^{-1} \ ?$$
Here's one way to approach it:
   (a) Write the following out, and check my algebra:
$$(x + \mathrm{d}x)-1 = \frac{1}{x + \mathrm{d}x} = \frac{1}{x\left(1 + \frac{\mathrm{d}x}{x}\right)} = \frac{1}{x}\frac{1}{\left(1 + \frac{\mathrm{d}x}{x}\right)}.$$
   (b) Assuming you believe that, substitute in $\frac{\mathrm{d}x}{x}$ for x in one of the formulas you found in question 4. Multiply through the brackets (but don't cancel the $(\mathrm{d}x)^2$ and so on just yet). This gives an expansion of $(x + \mathrm{d}x)^{-1}$. Based on your thinking in the last part of question 4, is this a case where the infinite algebra actually makes sense?
   (c) Now, use the rule $(\mathrm{d}x)^2 = 0$ to simplify the formula you found for $(x+\mathrm{d}x)^{-1}$. Use this to find the differential of $y = \frac{1}{x}$ in a different way than you did on Problem Set 2. Does it give the right answer??
8. (Open-ended) Try to make up some more formulas with infinite algebra, like the ones in question 4. How far can you take this??