How to find π

As I mentioned in class, I would like to try different ways of finding π. What does "finding" mean? For a start, it could mean giving a rough numerical estimate. Then, one could try to give improved numerical estimates. Something even better, as a next step, would be to find some pattern or system of estimates, so that you could, for example, easily program them in a computer. Even better would be an explicit formula, but that might be too much to hope for.

I am using π as an example, but at least some of the methods will extend to many other problems. In particular, since questions about π involve areas or arc lengths of circles, some of this might be relevant to questions on ellipses...

1. (a) As I started doing in class, take the area formula for a circle, $A = \pi r^2$, as a starting point to get a rough estimate for π. Enclose the circle in a square to get an overestimate ("upper bound"), and put a square inside the circle to get an underestimate ("lower bound"). Once you get both numbers, summarize your answer as a range that π must lie in, i.e. something like blank $< pi <$ blank, with the blanks filled in.
   (b) Use the circumference formula $C = 2\pi r$ as the starting point instead. Get a rough estimate for π using the same two squares. Do you get the same numbers as you did with area? If not, can you use this to narrow the range that you found for π?
2. Now, try to improve these estimates! You can use either the area strategy, or the circumference strategy, or both, whichever seems more promising to you. (If you're stuck where to begin, maybe think about how to figure out how far off those estimates from the last problem are.) Ideally, I'd like to find a narrower and narrower range in which the number π has to lie. See how close you can get!

Volumes

I want to introduce the concept of volumes by trying a 3-dimensional version of the previous problems on circles. My goal is to figure out the volume of a 3-dimensional solid sphere of known radius r. Equivalently, if I put that sphere into a cubical box, I want to know what proportion of that box it takes up.

3. Suppose we have a 3-dimensional solid sphere of known radius r.
   (a) Imagine putting the sphere into a cubical box (like how you put a circle into a square in a previous question). What is the volume of that box? Use that to give an upper bound (overestimate) for the volume of a sphere of radius r.
   (b) Imagine putting the sphere into a circular cylinder. What estimate do you get? What are advantages or disadvantages compared to putting the sphere in a box?
   (c) (Optional challenge) Now, imagine putting a cylinder, or a cubical box, inside the sphere, as large as possible. Can you figure out the dimensions of that cylinder or box? (You can do either cylinder or box, whichever you think will be better and/or easier.) Use that to give a lower bound (underestimate) for the volume of a sphere of radius r.
   (d) Putting everything that you worked out together, what would be your rough guess for the volume formula for a sphere of radius r?

4. I'm going to suggest a very sneaky strategy for finding the volume formula for a sphere. This sneaky strategy seems to have been first figured out by Newton (though there were precursors), and it will be used frequently throughout calculus. This question will give the first steps for this sneaky strategy; we'll finish it out next problem set.

   First, just to make things a little simpler, I am going to consider a half-sphere. I will try to find the volume of that, then double it in the end. (It's not obvious that will be simpler, but it's worth a try.)

   Second, I want to change this *static* problem of finding a volume into a *dynamical* problem, where something is changing. So, I am going to imagine that the half-sphere is a water tank, and I am going to gradually fill the tank with water.

   (a) Let's put the half-sphere on xyz coordinate axes, with the center of the sphere at the origin. I will think of z as the vertical axis, and the half-sphere will only be in the region $z \geq 0$. Draw a cross-section of the half-sphere from the side, just drawing the x-axis horizontally and the z-axis vertically. Assume that the sphere has a known radius r.

   (b) Suppose that the half-sphere is partially filled with water, to some depth h. Draw what this looks like on your cross-sectional diagram.

   (c) The water has a volume V. We can think of the volume V as being a function of h; we don't know the formula for V in terms of h. (I'm not asking anything here, just pointing out a step in my reasoning.)

   (d) Suppose now that I add a little more water, so that the depth h changes to $h + \mathrm{d}h$. If the original volume of water in the tank was V, then the volume of water increases to $V + \mathrm{d}V$. Draw the shape of additional volume $\mathrm{d}V$ that has been added.

   (e) Find that additional volume $\mathrm{d}V$, as a formula involving variables in the problem. (You can add variables if you need to, if you label them on your diagram.)

   (f) Now, write your formula for $\mathrm{d}V$ in terms of h, $\mathrm{d}h$, and r only (possibly also with some other constants).

   (g) Your formula in the previous part should allow you to find the derivative $\frac{\mathrm{d}V}{\mathrm{d}h}$ of the unknown function V depending on h. What can you do from there? How does this help with the original problem???

Chords and arcs of circles

As I mentioned in class, I would like to find a relation between the length of a chord of a circle and the corresponding arc. This will end up being relevant to finding π, to finding areas and arc lengths for ellipses, to the Kepler problem, and many other things besides.

Suppose I draw a circle of radius 1 on xy coordinate axes, with the center of the circle at the origin. Then I will draw a point (x, y) on the circle, and a line from the origin $(0, 0)$ to the point (x, y). I will call the arc length along the circle, from the point $(1, 0)$ to the point (x, y), by the letter s. See Figure 1 for distances, and Figure 2 for the corresponding coordinates. Note that the line from the origin $(0, 0)$ to the point (x, y) has a slope $m = y/x$, which can also be identified with the distance I showed on Figure 1 (since the base of that triangle is 1).

Here are my goals: I want to find

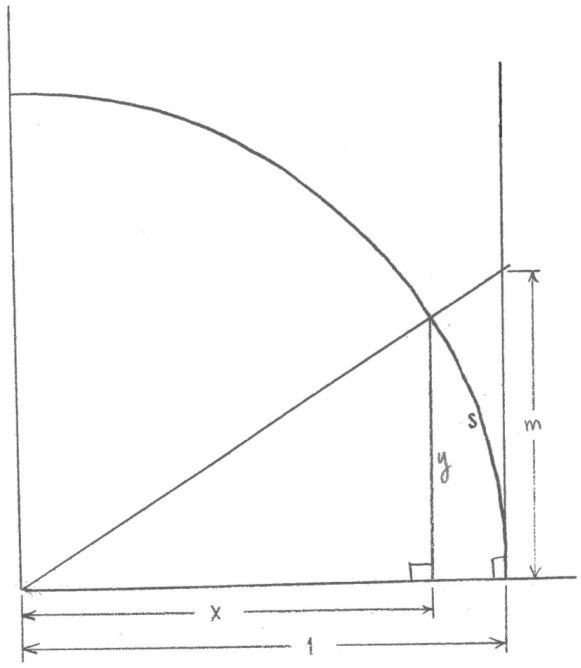


Figure 1

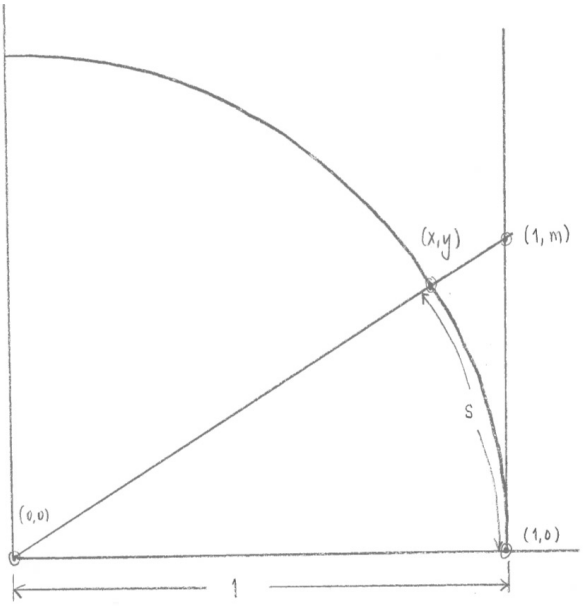


Figure 2

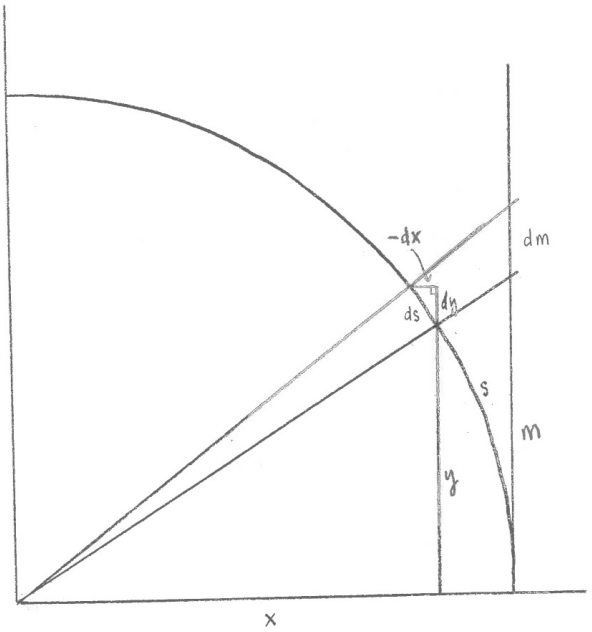


Figure 3

- A formula for the slope m in terms of the arc length s
- A formula for the arc length s in terms of the slope m
- A formula for the height y in terms of the arc length s
- A formula for the arc length s in terms of the height y
- A formula for the coordinate x in terms of the arc length s
- A formula for the arc length s in terms of the coordinate x

Note that these are solvable with trig functions; but I would like you to (temporarily!) forget about trig functions, because I want to (1) redefine trig functions in this way, and, more importantly, (2) find actual *formulas* for trig functions.

Here is what my strategy is going to be:

- Move the line a tiny amount. See Figure 3. (I have exaggerated the tiny amount so that it is readable in the diagram.)
- Then s changes to $s + \mathrm{d}s$, m changes to $m + \mathrm{d}m$, y changes to $y + \mathrm{d}y$, and x changes to $x + \mathrm{d}x$.
- Based on the diagram, I want to find formulas relating $\mathrm{d}m$ and $\mathrm{d}s$, $\mathrm{d}y$ and $\mathrm{d}s$, and $\mathrm{d}x$ and $\mathrm{d}s$.
- These relations of differential changes give me *differential equations* for the unknown functions m depending on s, y depending on s, and x depending on s.
- Then I'll try to solve these differential equations! (Although just knowing the relations of the differentials is itself useful.)

This problem will extend over several problem sets. We'll just get started here, with the following problems.

5. (a) Can you recall why the equation of the circle is $x^2 + y^2 = 1$? (You can take a certain theorem from elementary geometry for granted here...)
   (b) Find the differential of this equation to get a relationship between x, y, $\mathrm{d}x$, and $\mathrm{d}y$.
   (c) Recall from high school geometry that two lines with slopes m_1 and m_2 are perpendicular if $m_2 = -\frac{1}{m_1}$. (Optional: can you figure out why?) Use this fact, and what you just found about the differentials, to establish that the line segment from (x, y) to $(x + \mathrm{d}x, y + \mathrm{d}y)$ is perpendicular to the line from $(0, 0)$ to (x, y). (Optional: do you see how this is connected to a fact you learned in high school geometry?)
6. Look at the small triangle with sides $-\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}s$ in Figure 3. Let me make two notes, before asking you questions:
   - The segment with length $\mathrm{d}s$ isn't actually a straight line. I'm claiming that when $\mathrm{d}s$ is "infinitely small", that it can be treated as a straight line. (More precisely, the claim is that the difference of that segment from a straight line creates an error in our assumptions and formulas to follow, but that the error approaches zero in the limit as $\mathrm{d}s$ goes to zero.)
   - When we increase s slightly to $s + \mathrm{d}s$, the x-coordinate of our point actually *decreases*, so the corresponding $\mathrm{d}x$ is a *negative* number. This means that $-\mathrm{d}x$ is a positive number; that is why I have indicated the length of the top edge of the little triangle as $-\mathrm{d}x$ (length must be positive!).

   Here is the question: I would like you to find which angles are equal to which ones in this diagram, as much as possible. In particular, I would like you to relate the angles in the big right triangle (with sides of lengths x and y) to the angles in the little right triangle (with sides of lengths $-\mathrm{d}x$ and $\mathrm{d}y$). I'd also like you to figure out as many of the angles around the little triangle as you can (where "figure out" may mean to relate them to other angles in the diagram).

That's as far as I want to go now; in the next problem set, we'll use what you worked out to try to carry out the strategy I outlined.

Pascal's triangle and the binomial theorem

This section concerns Pascal's triangle and the binomial theorem. Suppose we expand powers of $(1 + x)$:

$$\begin{aligned}
(1+x)^1 &= 1 + x \\
(1+x)^2 &= 1 + 2x + x^2 \\
(1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\
(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4 \\
(1+x)^5 &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \\
(1+x)^6 &= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6 \\
&\vdots
\end{aligned}$$

The coefficients are called *binomial coefficients*. They can be put into a triangle called *Pascal's triangle*:

$$
\begin{array}{ccccccccc}
1 & & & & & & & \\
1 & 1 & & & & & & \\
1 & 2 & 1 & & & & & \\
1 & 3 & 3 & 1 & & & & \\
1 & 4 & 6 & 4 & 1 & & & \\
1 & 5 & 10 & 10 & 5 & 1 & & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

In Europe and America, we call it Pascal's triangle, because Pascal rediscovered and popularized it, but it has a much longer history; it seems to have been known to the Greeks and to Pingala in India about 200 BCE, and it was explicitly written as a triangle at least as far back as Al-Kharaji of Persia (popularized by Omar Khayyam, who was also a mathematician!) and Jia Xian of China in about 1000 CE.

In current European and American books it is often drawn as an isosceles triangle, but other authors (including Pingala and Newton) have drawn it as a right-angled triangle. The right-angled version will be clearer for what I want to do here.

These have the pattern that each entry is the sum of two of the entries on the previous line: the one directly above, and the one above and to the left. For example, in the last line I wrote, $15 = 5 + 10$, and $20 = 10 + 10$. (We'll see soon why this is true.)

7. (a) My triangle has a single "1" at the top. If you compare it to my table of powers of $(1 + x)$, there isn't a corresponding coefficient there. Can you make sense of that single "1" in terms of powers of $(1 + x)$?
   (b) Newton observed that you could add zeroes to the end of each of the rows written above, and it would still fit the rule for the triangle. In fact, you could think of each of the rows above continuing with infinitely many zero entries to the right, and the rule would still be obeyed. Check for yourself that this is true. (You don't have to write anything, just write the table with the zeroes filled in and check.)
   (c) Newton then thought of finding the row *above* the zeroth row, the one at top which contains only a single "1". Assume that this row above the zeroth row also starts with a 1, as the others do. Find the other entries of the row.
   (d) Can you interpret this new row that has been added?
   (e) How much further can you take this??